Scale-invariant regime in Rayleigh-Taylor bubble-front dynamics

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A statistical model of Rayleigh-Taylor bubble fronts in two dimensions is introduced. Float and merger of bubbles lead to a scale-invariant regime, with a stable distribution of scaled bubble radii and a constant front acceleration. The model is solved for a simple merger law, showing that a family of such stable distributions exists. The basins of attraction of each of these are mapped. The properties of the scale-invariant distributions for various merger laws, including a merger law derived from the Sharp-Wheeler model, are analyzed. The results are in good agreement with computer simulations. Finally, it is shown that for some merger laws, a runaway bubble regime develops. A criterion for the appearance of runaway growth is presented.

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I. INTRODUCTION

Rayleigh-Taylor (RT) instability occurs at an interface between a heavy fluid over a light fluid in a gravitational field [1]. Initial perturbations of the interface develop into a mixing zone, topped by rising bubbles of light fluid interleaved by falling spikes of heavy fluid. Experimental [2] and numerical [3,4,9] work indicate that the width of the mixing zone attains a constant acceleration independent of the initial perturbation.

The dynamics of the outer envelope of the mixing zone may be described in terms of bubbles rising and merging. An individual bubble (such as in a periodic array) rises with terminal velocity proportional to the root of its radius [1,5]. A small bubble adjacent to a large one is washed downstream, while its large neighbor expands, thus rising faster. This competition leads to an acceleration of the front [5].

This description of the RT mixing zone was pioneered by Sharp and Wheeler who proposed a model for bubble rise and merger [1]. The bubbles in this model are arranged along a line, and are characterized by their height h_i and radius r_i. The bubbles rise according to $dh_i/dt=c_1\sqrt{gr_i}$. Merger occurs when the height difference between adjacent bubbles exceeds c_2 times the radius of the smaller bubble. The surviving bubble radius is set to the sum of the radii of the two bubbles that merged. Thus the average velocity of the front increases with each merger. Numerical simulations of the Sharp-Wheeler (SW) model were performed by Gardner *et al.* [6] and Glimm and Li [7]. They find that the bubble front attains a constant acceleration independent of the initial bubble distribution.

Other bubble competition models have been subsequently developed. A model in which bubbles are represented by point sources in a potential flow was presented by Zufiraia [8]. Glimm *et al.* [9] developed a model in which the bubble rise parameters are fitted to hydrodynamic numerical simulations, and merger is given by a superposition of the single bubble and front evolution. Numerical simulations of these models show that the bubble fronts attain a constant acceleration, in agreement with experimental results.

Recently, Glimm and Sharp [10] showed that simplified dynamics of bubble-merger models may flow to a scale-invariant regime. The growth rate in such a model was calculated by Zhang [10], and was found to be in agreement with experimental results. However, the properties of the fixed distributions, the nature of the flow towards them, and the effects of different merger laws have not been studied.

In this article we describe a simple statistical model for the dynamics of a front of merging bubbles. This model uses only the bubble radii as dynamic variables (rather than the radii and heights as in the SW model). A mean-field evolution equation for the bubble-radius distribution is developed. In Sec. II we solve the model analytically for a simple merger law. We show that initial distributions flow, after a transient regime, into one of a family of scale-invariant distributions ("fixed points"). Each of these distributions is characterized by a different decay at large radii. We map the basins of attractions of each fixed point. In Sec. III we analyze the properties of the fixed-point distributions for various merger laws, including a merger law derived from the Sharp-Wheeler model. The results are compared to computer simulations. The effects of correlations between the radii of neighboring bubbles are discussed. In Sec. IV we show that for some merger rules, a regime of runaway bubbles appears. A criterion for the development of this regime is presented.

II. THE STATISTICAL MODEL

Previous models of bubble competition characterized each bubble with several dynamical variables such as its radius and height. In this article we present a statistical model of bubble competition which uses only the bubble radii as variables. This model includes the essential physics of the full problem, and is easier to analyze mathematically. We show, in Sec. III, that such a simplified model can even reproduce quantitative results of more elaborate models such as Sharp-Wheeler dynamics.

We consider an ensemble of bubbles of radii r_i. For simplicity we assume two-dimensional (2D) geometry: the bubbles are arranged along a line. Two adjacent bubbles of radii r and r' merge at a rate $\omega(r,r')$ forming a new bubble of radius $r+r'$ (thus the sum of all bubble radii is conserved). Here we make the important assumption that the bubble heights (and possibly other bubble parameters) may be averaged out of the merger rate. Hence we have a simple model in which the only dynamical variables are the bubble radii. The model may be visualized as points scattered along a line, each disappearing with a "half-life" dependent on the distance to its two nearest neighbors.

The physics of bubble merger is included in $\omega(r,r')$. We define $g(r,t)dr$ as the number of bubbles with a radius within dr of r at time t. An evolution equation for $g(r,t)$, neglecting correlations between near-neighbor radii (mean-field approximation), is

$$N(t)\frac{\partial g(r,t)}{\partial t}=-2g(r,t)\int_0^\infty g(r',t)\omega(r,r')dr' + \int_0^\infty g(r-r',t)g(r',t)\omega(r-r',r')dr' , \tag{1}$$

where $N(t)$ is the number of bubbles at time t,

$$N(t)=\int_0^\infty g(r)dr . \tag{2}$$

The first term on the right-hand side of Eq. (1) is the rate of merger of a bubble of radius r with its neighbors, and the second term is the rate of creation of bubbles of radius r by means of merger of two bubbles whose radii are r' and $r-r'$. Integrating Eq. (1) over r we get

$$\frac{dN(t)}{dt}=-\langle w\rangle N(t) , \tag{3}$$

where

$$\langle w\rangle=N(t)^{-2}\int_0^\infty\int_0^\infty g(r,t)g(r',t)\omega(r,r')dr\,dr' .$$

To understand the dynamics of this model, we begin by studying it with a simple merger law: $\omega(r,r')=\omega(t)$, i.e., the merger rate is independent of the bubble radii. A numerical simulation of this model is performed by scattering points along a line according to some initial distribution, and randomly removing points. It is easily seen that no correlation between the lengths of adjacent segments is produced, so the mean-field equation [Eq. (1)] is exact in this case.

The dynamics may be analytically solved for this case. We first switch to the Laplace transform of the radius distribution, $g(s,t)$, defined as

$$g(s,t)=\int_0^\infty \exp(-sr)g(r,t)dr .$$

We include any time dependence of ω in the time derivative of Eq. (1), and take $\omega=1$. Applying a Laplace transform to Eq. (1) we have

$$\frac{\partial g(s,t)}{\partial t}=-2g(s,t)+\frac{g^2(s,t)}{g(0,t)} , \tag{4}$$

whose solution is

$$g(s,t)=\frac{p(t)^2}{\{1/g(s,0)+[p(t)-1]/N(0)\}} , \tag{5}$$

where $p(t)=N(t)/N(t=0)=e^{-t}$ is the fraction of bubbles remaining, and $g(s,0)$ is the Laplace transform of the initial radius distribution. From Eq. (5), we see that the average radius is $\langle r(t)\rangle=\langle r(t=0)\rangle/p(t)$, as is true for any merger law since the sum of bubble radii is conserved. An initial δ distribution $g(r,t=0)=N(0)\delta(r-r_0)$ evolves into a series of δ functions with exponentially decreasing amplitudes:

$$g(r,t)=N(0)p(t)^2\sum_{n=1}^{\infty}[1-p(t)]^{n-1}\delta(r-nr_0) .$$

An initial distribution $g(r,t=0)=N(0)r\exp(-r)$ yields

$$g(r,t)=\frac{N(0)p(t)^2\exp(-r)\sinh[r\sqrt{1-p(t)}]}{\sqrt{1-p(t)}} . \tag{6}$$

Both these distributions flow to

$$g(r,t)=\frac{N(0)p(t)^2\exp[-r/\langle r(t)\rangle]}{\langle r(t)\rangle}$$

for large times. This behavior has been verified numerically (see Fig. 1).

We now ask which initial distributions flow into this asymptotic distribution. For this purpose we take a different approach to the solution of the model. At $t=0$, we regard the bubbles as points on a line with spacing distributed according to $P(r)=g(r,t=0)/N(t=0)$. We now consider a segment of length x after a fraction q of the points have been removed. If this segment is composed of n initial segments, $x=x_1+\cdots+x_n$, then for x large enough, its length distribution $\hat{P}_n(x)$ is given by the central limit distribution of the sum of n random variables of distribution $P(r)$. If $P(r)$ has a finite average r_0 and variance V, for instance, we get the Gaussian distribution

$$\hat{P}_n=\frac{1}{\sqrt{2\pi V}}\exp\left[-\frac{(x-nr_0)^2}{2nV}\right] . \tag{7}$$

The probability of finding a segment of length x is

$$P(x)=(1-q)\sum_{n=0}^{\infty}q^n\hat{P}_n(x) . \tag{8}$$

If $\hat{P}_n(x)$ is Gaussian [Eq. (7)], the sum may be evaluated by the saddle-point method, yielding $P(r) \sim \exp(-r/\langle r \rangle)$ for $r \gg r_0$, in agreement with the above results.

Initial distributions which fall off slower than x^{-3} at large x have different central limit distributions. The distributions of sums of independent random variables whose distributions fall off as $P(x) \sim x^{-(1+\mu)}$ for $0<\mu<2$ are known as Levy [11] distributions L_μ. For $0<\mu<1$, $\hat{P}_n(x) = L_\mu(x/n^{1/\mu})$, while for $1<\mu<2$ we have $\hat{P}_n(x) = L_\mu[(x-nr_0)/n^{1/\mu}]$. For an initial distribution which falls off as $cx^{-3/2}$, for instance, we have $L_{1/2}(u) = 2\pi c u^{3/2} \exp(-\pi c^2/u)$. Evaluating Eq. (8) for this case (by replacing the sum by an integral) we find that asymptotically, the segment distribution falls off as $P(x) \sim cx^{-3/2}$, the same as the initial distribution. This behavior is found for all distributions with $0<\mu<2$. We see, therefore, that there exists a family of asymptotic distributions for our model dynamics, each derived from a central limit distribution. The basin of attraction of each asymptotic distribution is the basin of attraction of the corresponding central limit distribution (see Ref. [11] for a complete mapping of these basins).

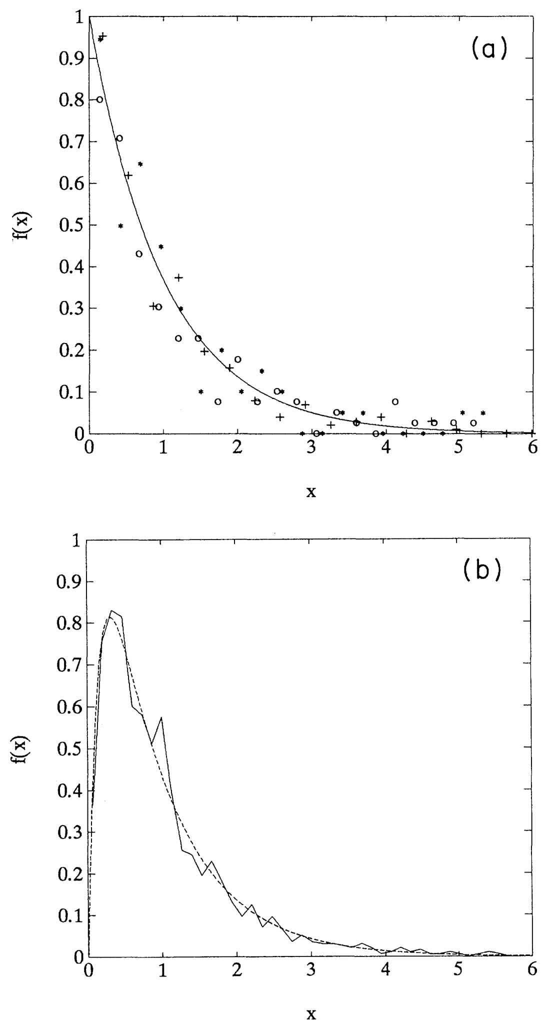

FIG. 1. (a) Radius distribution function for $\omega=1$ at $N/N_0=0.1$ (+), 0.05 (○), 0.02 (∗), plotted against $x=r/\langle r \rangle$. The line is the analytic result. Initial conditions were $N_0=3000$ bubbles with radii distributed uniformly [$U(0,1)$]. (b) Radius distribution for initial conditions $f(r,t=0)=r\exp(-r)$, when $N/N_0=\frac{1}{2}$. The dashed line is the analytic result [Eq. (6)].

The discussion above may be applied as a very simple model of RT bubble competition. Assume that the rate of merger is independent of bubble radii: the only dimensional parameter is the gravitational acceleration. The merger rate may thus be constructed from an acceleration G (proportional to the gravitational acceleration) and the average bubble radius: $\omega = \sqrt{G/\langle r(t) \rangle}$. This simple merger law falls within the framework of the above analysis, i.e., $\omega(r,r',t)=\omega(t)$. For this rule, it follows from Eq. (3) that

$$N(t) = N(0)(1+\tfrac{1}{2}\sqrt{G/r_0}\,t)^{-2} , \quad (9)$$

$$\langle r(t) \rangle = r_0(1+\tfrac{1}{2}\sqrt{G/r_0}\,t)^2 , \quad (10)$$

where r_0 is the initial average radius. Thus, after a transient regime ($t \gg 2\sqrt{r_0/G}$), the average radius attains a constant acceleration $\langle r(t) \rangle \to (G/4)t^2$, independent of the initial conditions. For all initial distributions with a finite variance, the radius distribution flows to its asymptotic form

$$g(r,t) = N(0)\frac{r_0^2}{G} t^{-4} \exp(-4r/Gt^2) . \quad (11)$$

Each bubble floats with a velocity proportional to the square root of its radius. The average velocity of the bubble front is thus proportional to the average root radius $\langle v(t) \rangle = b \langle \sqrt{r(t)} \rangle$. Asymptotically we find that the front moves at an acceleration of $\sqrt{\pi G}\,b/4$, independent of initial conditions.

For the simple merger law discussed in this section, we find that the dynamics leads to a scale-invariant regime. Actual bubble merger is strongly dependent on bubble parameters. Simulations of this model for various merger laws show that scale-invariant dynamics are attained for a large class of laws (see Sec. IV for examples of merger laws which do not reach a scale-invariant regime). In the next section we study the scale-invariant properties of our model for other merger laws.

III. SCALE-INVARIANT DISTRIBUTIONS

In this section we study the asymptotic regimes of the bubble-merger model for various merger laws. We assume that scale invariance develops after a transient regime.

When a scale-invariant regime is attained, the system parameters are independent of initial conditions, and depend on time only through the fraction of bubbles remaining (or, equivalently, on the average radius). We define the scaled radius distribution function f as follows:

$$g(r,t) = \frac{N(t)}{\langle r(t) \rangle} f(r/\langle r(t) \rangle) . \quad (12)$$

With this definition, using Eq. (2), we have

$$\int_0^\infty f(x)dx = 1 \ . \tag{13}$$

Inserting this definition into Eq. (1) we get an integro-differential equation for f:

$$\langle \omega \rangle \{ f(x) + \tfrac{1}{2} x f'(x) \} = f(x) \int_0^\infty f(x') w(x,x') dx' - \int_0^{x/2} f(x-x') f(x') \times w(x-x',x') dx' \ , \tag{14}$$

where $x = r / \langle r(t) \rangle$ and $\omega(x,x')$ is the scaled merger rate. Here we assume that ω may be scaled, namely, $\omega(r,r') = f(a)\omega(r/a, r'/a)$ for any a. For the simple merger law of Sec. II we set $\omega(x,x') = \omega_0$, and $f(x) = \exp(-x)$ indeed solves Eq. (14).

The constant acceleration attained by the bubble front is due to the appearance of the scale-invariant distribution. In RT problems, the merger rate scales as

$$\omega(r,r') = \omega \left[\frac{r}{\langle r \rangle}, \frac{r'}{\langle r \rangle} \right] \sqrt{\langle r \rangle g} \ .$$

Each bubble floats with a velocity $v = c\sqrt{rg}$. Thus, when a scale-invariant distribution is reached, the interface height y rises as $y = \alpha g t^2$, with

$$\alpha = \frac{c}{4} \langle \omega(x,x') \rangle_f \langle \sqrt{x} \rangle_f \ , \tag{15}$$

where the averages are over the scale-invariant distribution $f(x)$.

We now look at some asymptotic properties of $f(x)$.

A. Solutions near $x = 0$

If $f(x)$ is bounded (or weakly divergent) at $x = 0$, the convolution term in Eq. (14) is negligible for small x. In this limit, Eq. (14) takes the form

$$x f'(x) = 2 f(x) \{ \langle \omega(x) \rangle / \langle \omega \rangle - 1 \} \ ,$$

$$\langle w(x) \rangle = \int_0^\infty f(x') w(x,x') dx' \ .$$

For finite $\langle \omega(0) \rangle$ we have

$$f(x) \sim x^a, \quad \text{where } a = 2[\langle \omega(0) \rangle / \langle \omega \rangle - 1] \ . \tag{16}$$

For merger laws satisfying $\langle \omega(x) \rangle / \langle \omega \rangle \sim a / x^m$, with $m > 0$, we find

$$f(x) \sim \exp \left[-\frac{2a}{m x^m} \right] \ . \tag{17}$$

B. Sharp-Wheeler merger law

Realistic merger laws should include the following properties: unstable equilibrium for the case of an ensemble of bubbles of exactly the same radius [$\omega(r,r) = 0$], and diverging merger rates between bubbles of very different radii [$\omega(r,r') \to \infty$ for $r \gg r'$ and $r \ll r'$].

The Sharp-Wheeler model, described in the Introduction, has the above properties and has been previously studied [1,6,7]. We shall now derive a merger rate which allows us to study the SW dynamics in the framework of the present model. To eliminate the height variable, we make the simplifying assumption that all bubble heights are set equal after each merger. Indeed, the SW dynamics quickly eliminate adjacent bubbles with a large height difference, leading to a narrow height distribution [6,7]. This leads to a merger rate

$$\omega(r,r') = a \frac{|\sqrt{r} - \sqrt{r'}|}{\min(r,r')} \ , \tag{18}$$

with $a = c_1 \sqrt{g} / c_2$.

This merger rule leads to a scale-invariant distribution $f(x)$. At $x \sim 0$, $\langle \omega(x) \rangle \sim a \langle \sqrt{x} \rangle / x$. From Eq. (17) we thus expect $f(x) \sim \exp(-\beta / x)$ near $x = 0$.

We performed numerical simulations of the model with this merger rule. As in the preceding section, the simulation begins with points scattered on a line, according to some initial distribution of segment lengths. Each point is assigned a half-life which is the merger rate of its two neighboring segments: $\omega_i = \omega(r_i, r_{i+1})$. At each step in the simulation, a point is chosen at random with a weight $\omega_i / \sum_{k=1}^N \omega_k$ and erased. The half-lives are adjusted accordingly and a new point is chosen. We used initial uniform and exponential distributions of radii. In Fig. 2(a) we present numerical simulation results for the asymptotic radius distribution. At large x, $f \sim \exp(-\gamma x)$ (with $\gamma \sim 0.7$), as seen in Fig. 2(b). In Fig. 2(c) the behavior at small x is fitted to $\exp(-\beta / x)$ (we find $\beta \sim 0.3$). This result is in agreement with the above analysis. The asymptotic acceleration attained by the interface is given, in this model, by $\alpha = c_1^2 g \alpha_0 / c_2$, where

$$\alpha_0 = \frac{1}{2} \left\langle \frac{|\sqrt{x} - \sqrt{x'}|}{\min(x,x')} \right\rangle_f \langle \sqrt{x} \rangle_f \ .$$

Using the asymptotic distribution found above, we find $\alpha_0 \sim 0.54 \pm 0.01$, which is higher than the value $\alpha_0 \sim 0.48$ obtained from full simulations of the SW model [7]. In these simulations, however, an area-conservation rule was applied: after the bubble merger, the height of the surviving bubble was set so that its area was equal to the sum of the areas of the two bubbles before merger. The reduction in the surviving bubble height is $\Delta h = c_2 r_2^2 / (r_1 + r_2)$, with $r_2 < r_1$. When multiplied by the rate of merger, this reduction in height leads to a negative asymptotic acceleration of $\alpha' = -c_1^2 g \alpha_0' / c_2$, where

$$\alpha_0' = \frac{1}{2} \left\langle \frac{|\sqrt{x} - \sqrt{x'}| \min(x,x')}{x + x'} \right\rangle_f \left\langle \frac{|\sqrt{x} - \sqrt{x'}|}{\min(x,x')} \right\rangle_f \ . \tag{19}$$

We find $\alpha_0' \sim 0.05 \pm 0.01$. Thus the total acceleration, $\alpha_0 - \alpha_0' \sim 0.49 \pm 0.02$, agrees well with the results for the full SW model [7].

For this merger law, correlations between the radii of neighboring bubbles are generated [6]. Bubbles with very different radii quickly merge, while those with similar radii survive. Hence a positive correlation between radii is expected. In Fig. 3 we plot the correlation

$$c(k) = \langle (r_i - \langle r \rangle)(r_{i+k} - \langle r \rangle) \rangle / (\langle r^2 \rangle - \langle r \rangle^2)$$

with the first four neighbors. Significant correlations are attained with the nearest neighbor, $c(1) \sim 0.2$. This value is in agreement with full simulations of the SW model [6].

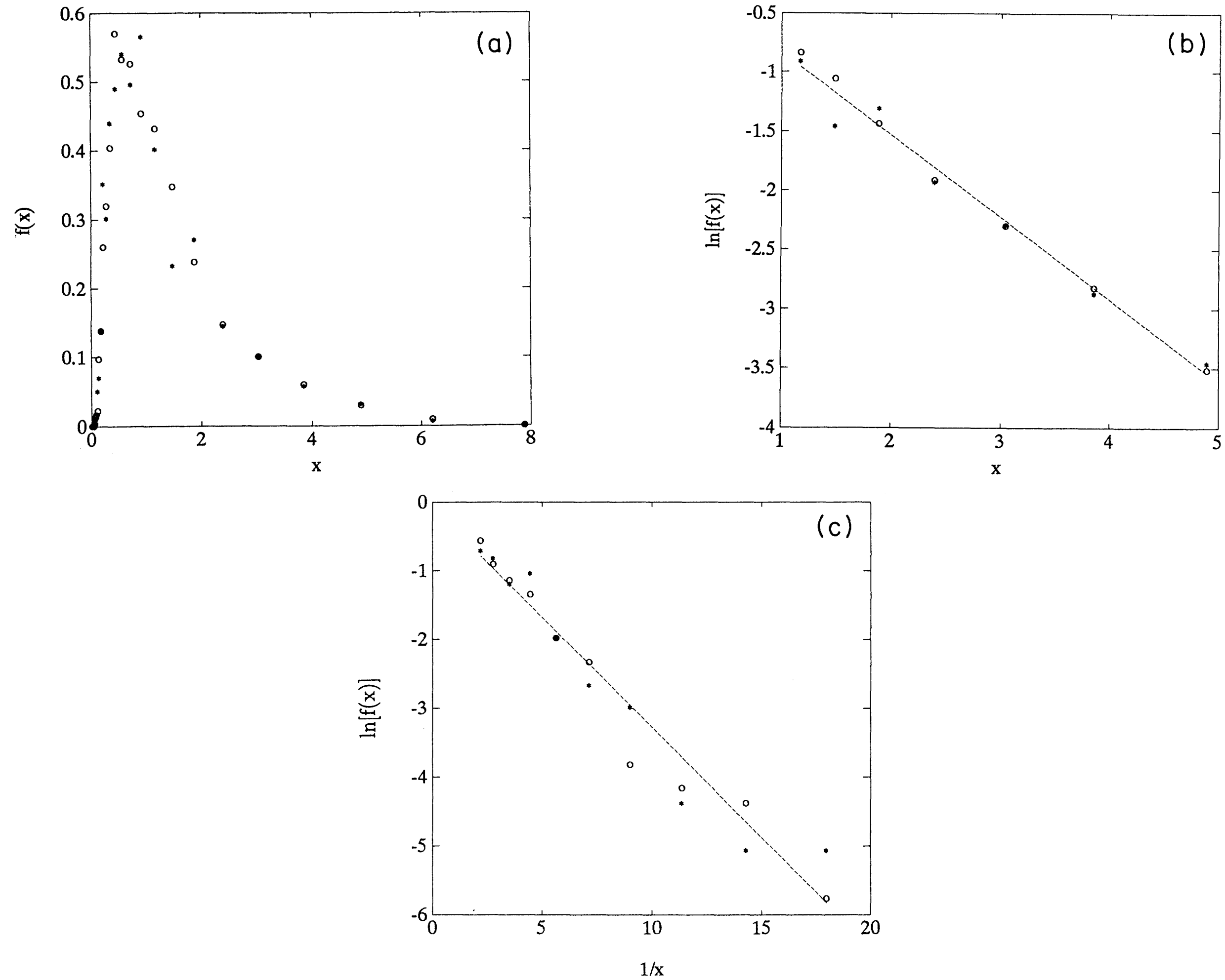

FIG. 2. (a) Scaled radius distribution function for Sharp-Wheeler merger rule. Stars are at 5% and ○'s at 2.5% of initial 30 000 bubbles [the initial distribution is $U(0,1)$]. (b) Semilog plot $f(x)$ vs x. (c) Semilog plot of $f(x)$ vs $1/x$ shows that $f(x) \sim \exp(-\beta/x)$ for small x.

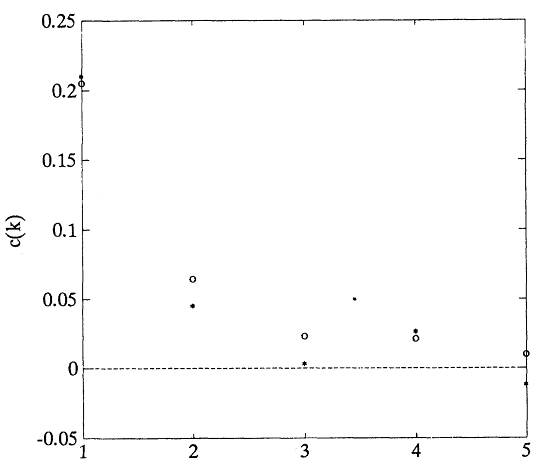

FIG. 3. Correlations between neighboring bubble radii for the Sharp-Wheeler merger rule. The simulation started with 30 000 bubbles, distributed uniformly, $U(0,1)$. Correlations were measured when 5% (*), 2.5% (○) of the bubbles remained.

Scale-invariant growth appears in many dynamical models [12]. In the next section we demonstrate a different type of growth regime.

IV. RUNAWAY BUBBLE REGIMES

For some merger laws, the model dynamics evolve into a regime where a very large "runaway" bubble dominates the merger process. In this section we study this phenomenon and give a criterion for determining when it will develop.

In order to demonstrate runaway growth, we consider the rule $\omega(r,r')=k(r-r')^2$. We performed numerical simulations of this rule for various initial conditions. Simulation shows (Fig. 4) that for a pulse-shaped initial distribution the mergers are uniformly distributed over the sample for a time, until runaway bubbles appear. We find that for narrower initial distributions, the onset of runaway growth appears earlier (i.e., at a larger fraction of bubbles surviving) in a given sample size.

Physically it is clear why runaway occurs earlier in narrow distributions. In narrow distributions a single merger produces a bubble whose radius is about twice the

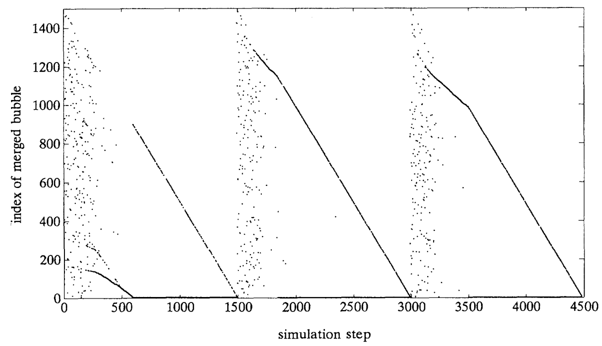

FIG. 4. Index of bubble merged vs number of simulation steps, for three runs with an initial pulse distribution $U(0,1)$. Uniformly dispersed mergers are replaced by runaway bubbles (streaks of mergers with adjacent indices). Note that due to the cyclic boundary conditions, the bubbles at the two ends of the front are equivalent.

average radius. The merger law gives strong preference to this large bubble. It merges with ever greater probability, regardless of its neighbor's size. Thus a runaway regime is attained very early, in which large bubbles grow in a field of small ones. In terms of the variable $x=r/\langle r\rangle$, the initial distribution is pushed toward $x=0$. For distributions with many large bubbles, however, the initial mergers occur when large bubbles are adjacent to small ones, small bubbles are eliminated, and a scale-invariant distribution function may be attained for a time. Runaway growth, however, ultimately appears.

We now present a criterion for the development of runaway growth. Suppose that at time $t=0$ runaway growth begins. A single large bubble continually expands in a field of smaller bubbles whose radius distribution is fixed. The runaway bubble thus attains a radius of $r_{\max}\sim(N_0-N)r_0$ where r_0 is the average radius of the bubble field, N is the number of bubbles remaining, and N_0 is the initial number of bubbles. For runaway growth to continue, the merger rate of the runaway bubble must overshadow the rest of the mergers

$$\omega_{\max} \gg \sum_{i=1}^{N} \omega_i = N\langle\omega\rangle_{\text{field}} , \tag{20}$$

where the sum is taken over the rest of the bubbles. For $\omega\sim(r-r')^2$, we find that the fraction of bubbles remaining, $p=N/N_0$, must satisfy $p/(1-p)^2 \ll N_0 r_0^2/2V$ where V is the variance of the initial distribution. Thus runaway behavior occurs for all initial distributions (with a finite variance).

On the other hand, for a law like the SW merger law, runaway growth cannot take place: $\omega_{\max}\sim\sqrt{N_0-N}/\sqrt{r_0}$ and for this to dominate the mergers we must have $p/\sqrt{1-p} \ll \langle\omega\rangle_{\text{field}}\sqrt{r_0}/\sqrt{N_0}$: Runaway may occur only when $p\ll 1/\sqrt{N_0}$, a negligible fraction of the evolution time for large bubble systems.

From Eq. (20) it is evident that runaway growth is intrinsic only for merger laws which obey the following criterion:

$$\int_0^\infty \frac{\omega(N_0\langle r\rangle(1-p),r)}{N_0\langle r\rangle} f(r)dr \gg p\frac{\langle\omega\rangle}{\langle r\rangle} ,$$

where p is the fraction of bubble merged, $f(r)$ is the background bubble-radius distribution, and $\langle r\rangle$ and $\langle\omega\rangle$ are the average background radius and merger rate when runaway growth began. A sufficient condition for runaway growth in large systems is, therefore,

$$\lim_{x\to\infty} \frac{\omega(x,x')}{x} > 0 \tag{21}$$

for all x', while laws where

$$\lim_{x\to\infty} \frac{\omega(x,x')}{x} = 0$$

uniformly for all x' are stable.

V. CONCLUSIONS

In this paper we considered a statistical model of Rayleigh-Taylor bubble fronts. This model includes only the bubble radii as variables, and bubble rise velocity and merger rate as parameters. Bubble merger continuously creates bigger bubbles, increasing the front velocity. In Sec. II we presented mean-field equations for this model, and solved them for a simple merger law. The dynamics evolve to a scale-invariant regime, and thus to a constant front acceleration. The initial radius distribution flows to one of a family of stable distributions characterized by a different decay at infinity. In Sec. III we found some asymptotic properties of the scale-invariant distribution. We showed that the Sharp-Wheeler model, which has bubble heights as well as radii as variables, may be successfully reduced to our model with an averaged merger law. Runaway growth was demonstrated in Sec. IV. Conditions on the merger law, sufficient for runaway growth to occur, have been given.

We note that in the mean-field approximation, our bubble-merger model is equivalent to the Smoluchowsky model of coagulating systems [12], in which clusters of particles (such as polymers in a solution) merge to form larger clusters. Runaway growth is analogous to the appearance of an infinite cluster in these coagulating systems (the sol-gel transition).

The existence of scale-invariant regimes, with families of fixed distributions, occurs in several growth problems [13]. Thus it is probably a much more general phenomenon.

Finally, in the context of RT front propagation, we note that other merger models may be treated in the framework of the present model, as demonstrated for the Sharp-Wheeler merger law. The simple merger rule of Sec. II may serve as a first approximation to the behavior of bubble fronts with various initial conditions. The only parameter in this approximation is the average merger rate. For any other merger rule, the front acceleration in the scale-invariant regime is given by Eq. (15). The present model explains the experimentally observed constant front acceleration in a rather simple way. The approach to a scale-invariant distribution with an immense basin of attraction explains why the front acceleration is independent of initial conditions.

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